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Gauss maps with nontrivial separable degree in positive characteristic

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Abstract

For a projective variety of dimension n in a projective space $\mathbb{P}^N$ defined over an algebraically closed field, the Gauss map is the rational map of the variety to the Grassmannian of n-planes in $\mathbb{P}^N$, mapping a smooth point to the embedded tangent space to the variety at the point. The purpose here is to give three examples of Gauss maps with separable degrees greater than one onto their images in positive characteristic: (1) a smooth variety with Kodaira dimension $\kappa < n$; (2) a normal variety of general type with only isolated singularities; (3) $\mathbb{P}^n$, whose image of the Gauss map is a normal variety of general type.

MSC: 14N05; 14E25

1. Introduction

Let X be a projective variety of dimension n over an algebraically closed field K of positive characteristic p. Let $\iota : X \to \mathbb{P}^N$ be an embedding. The Gauss map $\iota^{(1)}$ of ι is the rational map of X to the Grassmannian $\mathbb{G} := \mathrm{Grass}(n, \mathbb{P}^N)$ of n-planes in $\mathbb{P}^N$ mapping a smooth point $x \in X$ to the embedded tangent space to X at x in $\mathbb{P}^N$, defined over the regular locus $\mathrm{Reg}(X)$ on X as a morphism. The purpose of this paper is to give examples of projective varieties X with embeddings $\iota : X \to \mathbb{P}^N$ whose Gauss maps have nontrivial separable degrees onto their images in three cases: (1) X is a smooth variety with Kodaira dimension $\kappa < n$; (2) X is a normal variety of general

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type with only isolated singularities; (3) $X \cong \mathbb{P}^n$, whose image $\iota^{(1)}(\mathbb{P}^n)$ is a normal variety of general type.

Our results here are the following.

Theorem 1.1. *Assume that* $p > 0$. *For a given integer* $n \geq 2$ *and* $\kappa = -\infty, 0, 1, \ldots,$ *or* $n - 1$, *there exist a smooth projective variety* X *of dimension* n *with Kodaira dimension* κ *and an embedding* $\iota : X \to \mathbb{P}^N$ *whose Gauss map* $\iota^{(1)}$ *has nontrivial separable degree onto its image.*

Theorem 1.2. *Assume that* $p > 2$. *For a given integer* $n \geq 2$, *there exist a normal projective variety* X *of dimension* n *and Kodaira dimension* n *with only isolated singularities, and an embedding* $\iota : X \to \mathbb{P}^N$ *whose Gauss map* $\iota^{(1)} : \mathrm{Reg}(X) \to \mathbb{G}$ *has nontrivial separable degree onto its image.*

Theorem 1.3. *Assume that* $p > 2$. *For a given integer* $n \geq 2$, *there exists an embedding* $\iota : \mathbb{P}^n \to \mathbb{P}^N$ *such that the image* $\iota^{(1)}(\mathbb{P}^n)$ *of the Gauss map* $\iota^{(1)}$ *is a normal variety of general type.*

Due to a result of Zak [11, (I.2.8)], it is well-known that the Gauss map of a smooth variety not linear is a finite morphism, moreover if the characteristic of the base field is zero, it is birational onto its image. But if the characteristic is positive, the birationality is no longer true in general. In fact, as pointed out by A. Wallace in 1956, the Gauss map of the Fermat hypersurface of degree $p + 1$ is the K-Frobenius. In several cases, the structure of Gauss maps in positive characteristic is known. For dimension one, Kaji [2,3] showed that a smooth curve of genus greater than one and a supersingular elliptic curve always have purely inseparable Gauss maps for any embeddings, and that an ordinary elliptic curve and $\mathbb{P}^1$ have embeddings whose Gauss maps are inseparable but not purely inseparable. The embedding of $\mathbb{P}^1$ is given also by Rathmann [10, (2.13)]. For higher dimensional X, several results are known [4,5,8]. In particular, for smooth surfaces X with an embedding $\iota : X \to \mathbb{P}^N$, if the maximal destabilizing sheaf of the tangent bundle T_X has negative degree with respect to the adjoint bundle $K_X \otimes \mathcal{O}_X(3)$, then $\iota^{(1)}$ is purely inseparable [8, (1.2)]. From these results on curves and surfaces, it is natural to ask whether there exists an embedding of a smooth projective surface of positive Kodaira dimension whose Gauss map is inseparable but not purely inseparable. Our results above give a partial answer to this question.

Unless otherwise mentioned, we work over an algebraically closed field K of positive characteristic p.

2. Smooth varieties of special type

Theorem 1.1 is a consequence of the following theorem.

Theorem 2.1. *Let Z be a smooth projective variety with trivial cotangent bundle. Let Y be a smooth projective variety of dimension n having a smooth morphism $\mu : Y \to Z$ with a section. Assume that Z has a primitive lp-torsion line bundle $\mathscr{L}$ for some positive integer l not divisible by p, i.e., $\mathscr{L}^{\otimes lp} \cong \mathcal{O}_Z$ but $\mathscr{L}^{\otimes i} \ncong \mathcal{O}_Z$ for any i $(0 < i < lp)$. Then there exist a smooth projective variety X of dimension n, a finite morphism $\phi : X \to Y$, and an embedding $\iota : X \to \mathbb{P}^N$ satisfying the following properties*:

(1) *The canonical bundle ω_X is isomorphic to $\phi^*(\omega_Y \otimes \mu^*\mathscr{L}^{\otimes lp-1})$.*

(2) *The separable and inseparable degrees of ϕ are l and p, respectively.*

(3) *The Gauss map $\iota^{(1)} : X \to \mathbb{G} := \mathrm{Grass}(n, \mathbb{P}^N)$ of ι is the composite of ϕ and a morphism $Y \to \mathbb{G}$.*

(4) *Unless $p = 2$ and $n = 2$, the morphism $Y \to \mathbb{G}$ is an embedding.*

Proof of Theorem 1.1. Let Z be an ordinary elliptic curve, and hence Z has a primitive lp-torsion line bundle for every integer l not divisible by p (see [7, III, Section 15]). Let W be a smooth projective variety of dimension $n-1$ and Kodaira dimension κ. Set $Y = W \times Z$ and let μ be the second projection $Y \to Z$. By Theorem 2.1, there exists a smooth projective variety X of Kodaira dimension κ, a finite morphism $\phi : X \to Y$, and an embedding $\iota : X \to \mathbb{P}^N$ with the properties. By (2) and (3), $\iota^{(1)}$ has separable degree l or more. $\square$

The proof of Theorem 2.1 proceeds in four steps.

Step 1. (Construction of X): *Set $L := \mu^*\mathscr{L}$. By $L^\vee$ we denote the dual of L. Then there exist a smooth projective variety X of dimension n, a finite morphism $\phi : X \to Y$ of separable and inseparable degrees l and p, respectively, an $\mathcal{O}_Y$-linear $dL : (L^\vee)^{\otimes lp} \to \Omega^1_Y$, and an exact sequence*

$$0 \to \phi^*(L^\vee)^{\otimes lp} \xrightarrow{\phi^*(dL)} \phi^*\Omega^1_Y \xrightarrow{d\phi} \Omega^1_X \to \phi^*(L^\vee) \to 0. \tag{2.1.1}$$

Proof. First note that L is a primitive lp-torsion line bundle on Y, since $\mathscr{L}$ is so and μ has a section. Thus we have an $\mathcal{O}_Y$-algebra $\mathscr{A} := \bigoplus_{i=0}^{lp-1}(L^\vee)^{\otimes i}$ with $H^0(Y, \mathscr{A}) = H^0(Y, \mathcal{O}_Y) = K$, and hence a connected closed subscheme $X = \mathbf{Spec}_Y\mathscr{A}$ of the (geometric) line bundle $\mathbb{V} := \mathbb{V}_Y(L^\vee)$ with finite morphism $\phi : X \to Y$ (see [1, II. Exercise 5.17]).

Let $\{U_i\}$ be an affine open covering of Y such that $L|U_i$ is trivial with a basis τ_i and such that its transition functions $\{g_{ij}\}$ with $g_{ij}\tau_j = \tau_i$ satisfy $g_{ij}^{lp} = f_j/f_i$ for some $f_i \in \mathcal{O}^*_{U_i}$. For the dual $z_i \in L^\vee|U_i$ of τ_i, the isomorphism $(L^\vee)^{\otimes lp} \to \mathcal{O}_Y$ on U_i maps z_i^{lp} to f_i, since $z_i^{lp} = (g_{ij}^{-1})^{lp} z_j^{lp} = (f_i/f_j) z_j^{lp}$. Hence the open subset $\phi^{-1}(U_i)$ of X is defined by $z_i^{lp} - f_i$ in $\mathbb{V}_{U_i}(L^\vee|U_i)$. Consequently, ϕ has separable and inseparable degrees l and p, respectively.

Since $\tau_i^{lp} \otimes df_i = (g_{ij}^{-1})^{lp}\tau_i^{lp} \otimes df_j = \tau_j^{lp} \otimes df_j$, a set of local sections $\{\tau_i^{lp} \otimes df_i\}$ defines a global section of $L^{lp} \otimes \Omega^1_Y$ and hence the map $dL : (L^\vee)^{\otimes lp} \to \Omega^1_Y$ mapping z_i^{lp} to df_i. Then we see that dL is nonzero at $y \in Y$ if and only if X is smooth at every $x \in X$ with $\phi(x) = y$. Moreover dL is a nonzero map. Indeed, if $dL = 0$ then

$df_i = 0$, and hence $f_i = h_i^p$ for some $h_i \in \mathcal{O}_{U_i}$. Thus $g_{ij}^l = h_i/h_j$, which means $L^{\otimes l} \cong \mathcal{O}_Y$, contradiction.

To get (2.1.1), first note that $\phi^*(dL)$ is injective since it is nonzero and $\phi^*(L^\vee)^{\otimes lp}$ is of rank one. Also note that $\Omega^1_{X/Y} \cong \Omega^1_{\mathbb{V}/Y}|X \cong \phi^*(L^\vee)$, since the natural map $\mathscr{I}/\mathscr{I}^2 \to \Omega^1_{\mathbb{V}/Y}|X$ is the zero map for the ideal sheaf $\mathscr{I}$ of X in $\mathbb{V}$. Moreover, from the canonical diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{I}/\mathscr{I}^2 & \to & \Omega^1_{\mathbb{V}}|X & \to & \Omega^1_X & \to & 0 \\ & & & & \| & & & & \\ 0 & \to & \phi^*\Omega^1_Y & \to & \Omega^1_{\mathbb{V}}|X & \to & \Omega^1_{\mathbb{V}/Y}|X & \to & 0, \end{array}$$

we have two maps $\alpha : \mathscr{I}/\mathscr{I}^2 \to \phi^*\Omega^1_Y$ and $\beta : \Omega^1_X \to \Omega^1_{\mathbb{V}/Y}|X$. Under the isomorphism $\mathscr{I}/\mathscr{I}^2 \cong \phi^*(L^\vee)^{\otimes lp}$ due to $g_{ij}^{lp}(z_i^{lp} - f_i) = z_j^{lp} - f_j$, we have $\alpha = \phi^*(dL)$ since $d(f_i - z_i^{lp}) = df_i$. By the snake lemma, we have $\mathrm{Coker}(\phi^*(dL)) \cong \mathrm{Ker}(\beta)$, and hence (2.1.1).

Since Ω^1_Z is trivial, $(\mathscr{L}^\vee)^{\otimes lp}$ is a subbundle of Ω^1_Z by the nonzero $\mathcal{O}_Z$-linear map $d\mathscr{L}$ obtained from Z and $\mathscr{L}$ in the same way. Since μ is smooth, $(L^\vee)^{\otimes lp}$ is a subbundle of Ω^1_Y by dL. Therefore X is smooth. □

Step 2. (Embedding of X in projective space): *Let M be a very ample line bundle on Y such that $L \otimes M^{\otimes p}$ is also very ample. Let $s_0, \ldots, s_r$ be global sections generating M and linearly independent over K. Let V be the K-vector subspace of $H^0(Y, M^{\otimes p} \oplus L \otimes M^{\otimes p})$ generated by $\{s_0^{\otimes p}, \ldots, s_r^{\otimes p}\}$ and a K-basis $\{v_0, \ldots, v_m\}$ of $H^0(Y, L \otimes M^{\otimes p})$. Then the morphism λ from the projective space bundle $\mathbb{P}_Y(M^{\otimes p} \oplus L \otimes M^{\otimes p})$ to the projective space $\mathbb{P}^N := \mathbb{P}(V)$ with $N = r + m + 1$, defined by V, is an embedding except over the ∞-section $\mathbb{P}_Y(M^{\otimes p})$. In particular, if we denote the composite*

$$X \to \mathbb{V}_Y(L^\vee) \to \mathbb{P}_Y(L^\vee \oplus \mathcal{O}_Y) \cong \mathbb{P}_Y(M^{\otimes p} \oplus L \otimes M^{\otimes p}) \xrightarrow{\lambda} \mathbb{P}^N$$

by $\imath : X \to \mathbb{P}^N$, then $\imath$ is an embedding.

This follows from the next lemma.

Lemma 2.2. *Let Y be a projective variety of dimension n and G a rank-two vector bundle fitting into an exact sequence $0 \to L \to G \to \mathcal{O}_Y \to 0$ with a line bundle L. Let M be a very ample line bundle on Y such that $L \otimes M^{\otimes p}$ is very ample and such that the induced map $H^0(G \otimes M^{\otimes p}) \to H^0(M^{\otimes p})$ is surjective. Let $s_0, \ldots, s_r$ be global sections generating M and linearly independent over K. Let $u_0, \ldots, u_r \in H^0(G \otimes M^{\otimes p})$ be lifts of $s_0^{\otimes p}, \ldots, s_r^{\otimes p} \in H^0(M^{\otimes p})$. Let V be the K-vector subspace of $H^0(G \otimes M^{\otimes p})$ generated by $u_0, \ldots, u_r$ and $H^0(L \otimes M^{\otimes p})$. Then the morphism $\lambda : \mathbb{P}_Y(G \otimes M^{\otimes p}) \to \mathbb{P}(V)$ defined by V is an embedding except over the ∞-section $\mathbb{P}_Y(M^{\otimes p})$.*

Proof. Let $v_0, \ldots, v_m$ be a K-basis of $H^0(L \otimes M^{\otimes p})$ so that $\{u_0, \ldots, u_r, v_0, \ldots, v_m\}$ is one of V. Set $G' = G \otimes M^{\otimes p}$. To see that λ is injective, let P_1 and P_2 be two distinct points of $\mathbb{P}_Y(G') \setminus \mathbb{P}_Y(M^{\otimes p})$ and set $y_i = \pi(P_i)$, where $\pi : \mathbb{P}_Y(G') \to Y$ is the projection. Set $P_i = [a_i, 1] \in \mathbb{P}(G' \otimes K(y_i))$ $(i = 1, 2)$ under isomorphisms $G' \otimes K(y_i) \cong$

$(M^{\otimes p} \oplus L \otimes M^{\otimes p}) \otimes K(y_i)$, where $K(y_i)$ denotes the residue field at y_i. If $y_1 \neq y_2$, the evaluation map

$$V \xrightarrow{\alpha} G' \otimes (K(y_1) \oplus K(y_2)) \cong (M^{\otimes p} \oplus L \otimes M^{\otimes p}) \otimes (K(y_1) \oplus K(y_2)) \xrightarrow{\beta} K(P_1) \oplus K(P_2)$$

is given by matrices

$$\alpha = \begin{bmatrix} A & 0 \\ * & B \end{bmatrix}, \qquad \beta = \begin{bmatrix} a_1 & 0 & 1 & 0 \\ 0 & a_2 & 0 & 1 \end{bmatrix}$$

for canonical bases of $(M^{\otimes p} \oplus L \otimes M^{\otimes p}) \otimes K(y_i)$ and $K(P_i)$ and for that of V. Here A is a $2 \times (r+1)$-matrix, and B is a $2 \times (m+1)$-matrix of rank 2 since $L \otimes M^{\otimes p}$ is very ample. Thus, the evaluation map is surjective, and hence $\lambda(P_1) \neq \lambda(P_2)$. On the other hand, if $y_1 = y_2$, the evaluation map

$$V \xrightarrow{\alpha} G' \otimes K(y_1) \cong (M^{\otimes p} \oplus L \otimes M^{\otimes p}) \otimes K(y_1) \xrightarrow{\beta} K(P_1) \oplus K(P_2)$$

is given by matrices

$$\alpha = \begin{bmatrix} \boldsymbol{a} & 0 \\ * & \boldsymbol{b} \end{bmatrix}, \qquad \beta = \begin{bmatrix} a_1 & 1 \\ a_2 & 1 \end{bmatrix}$$

for the bases as before. By the same reason as above, $\boldsymbol{a}$ and $\boldsymbol{b}$ are a nonzero $1 \times (r+1)$-matrix and a $1 \times (m+1)$-matrix, and consequently α is of rank 2. Since $a_1 \neq a_2$, the evaluation map is surjective, and hence $\lambda(P_1) \neq \lambda(P_2)$.

To see that λ is an immersion at $P \in \mathbb{P}_Y(G') \setminus \mathbb{P}_Y(M^{\otimes p})$, set $y = \pi(P) \in Y$. For $\mathbb{P} := \mathbb{P}_Y(G')$ and the maximal ideal m_P at P, as a K-vector space, $\mathcal{O}_{\mathbb{P}}(1) \otimes (\mathcal{O}_{\mathbb{P}}/m_P^2)$ is naturally isomorphic to $M^{\otimes p} \otimes K(y) \oplus (L \otimes M^{\otimes p}) \otimes (\mathcal{O}_Y/m_y^2)$. Thus the evaluation map $V \to \mathcal{O}_{\mathbb{P}}(1) \otimes (\mathcal{O}_{\mathbb{P}}/m_P^2)$ is the map

$$V \to M^{\otimes p} \otimes K(y) \oplus (L \otimes M^{\otimes p}) \otimes (\mathcal{O}_Y/m_y^2)$$

given by a matrix

$$\begin{bmatrix} \boldsymbol{a} & 0 \\ * & B \end{bmatrix}$$

for the basis V and for canonical bases of $M^{\otimes p} \otimes K(y)$ and $(L \otimes M^{\otimes p}) \otimes (\mathcal{O}_Y/m_y^2)$. Since $\boldsymbol{a}$ is a nonzero $1 \times (r+1)$-matrix and B is a nonzero $(n+1) \times (m+1)$-matrix of rank $n+1$, the evaluation map is surjective, and hence λ is an immersion at P. $\square$

Step 3. The Gauss map $\iota^{(1)} : X \to \mathbb{G}$ of $\iota : X \to \mathbb{P}^N = \mathbb{P}(V)$ factors through $\phi : X \to Y$.

Proof. Let $P_X^1(\mathcal{O}_X(1))$ be the bundle of principal parts of $\mathcal{O}_X(1) := \iota^*\mathcal{O}_{\mathbb{P}^N}(1)$ of the first order on X with left $\mathcal{O}_X$-module structure, and let a^1 be the natural surjection $V \otimes \mathcal{O}_X \to P_X^1(\mathcal{O}_X(1))$ (see for example [9, Section 6]). We have only to show that a^1 comes from Y since $\iota^{(1)}$ is obtained from a^1 by the universality of the Grassmannian $\mathbb{G}$. First note that $\mathcal{O}_X(1) \cong \phi^*(L \otimes M^{\otimes p})$, since $\mathbb{P}(L^\vee) \in |\mathcal{O}_{\mathbb{P}(L^\vee \oplus \mathcal{O})}(1)|$ does not intersect X and hence $\mathcal{O}_{\mathbb{P}(L^\vee\oplus\mathcal{O})}(1)|X \cong \mathcal{O}_X$. Also note that $P^1_X(\mathcal{O}_X(1))$ fits into the exact sequence

$$0 \to \phi^*\left(\frac{\Omega^1_Y}{(L^\vee)^{\otimes lp}}\right) \otimes \mathcal{O}_X(1) \to P^1_X(\mathcal{O}_X(1)) \to P^1_{X/Y}(\mathcal{O}_X(1)) \to 0 \tag{2.3.1}$$

which follows from (2.1.1) and from the natural diagram

$$\begin{array}{ccccccccc} 0 & \to & \Omega^1_X \otimes \mathcal{O}_X(1) & \to & P^1_X(\mathcal{O}_X(1)) & \to & \mathcal{O}_X(1) & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \Omega^1_{X/Y} \otimes \mathcal{O}_X(1) & \to & P^1_{X/Y}(\mathcal{O}_X(1)) & \to & \mathcal{O}_X(1) & \to & 0. \end{array}$$

On the other hand, for $\mathbb{P} := \mathbb{P}_Y(M^{\otimes p} \oplus L \otimes M^{\otimes p})$, from the isomorphism $\Omega^1_{X/Y} \cong \Omega^1_{\mathbb{V}/Y}|X$ in Step 1 and the canonical diagram

$$\begin{array}{ccccccccc} 0 & \to & \Omega^1_{\mathbb{P}/Y} \otimes \mathcal{O}_{\mathbb{P}}(1)|X & \to & P^1_{\mathbb{P}/Y}(\mathcal{O}_{\mathbb{P}}(1))|X & \to & \mathcal{O}_{\mathbb{P}}(1)|X & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \Omega^1_{X/Y} \otimes \mathcal{O}_X(1) & \to & P^1_{X/Y}(\mathcal{O}_X(1)) & \to & \mathcal{O}_X(1) & \to & 0, \end{array}$$

we obtain $P^1_{X/Y}(\mathcal{O}_X(1)) \cong P^1_{\mathbb{P}/Y}(\mathcal{O}_{\mathbb{P}}(1))|X$. Consequently $P^1_{X/Y}(\mathcal{O}_X(1)) \cong \phi^*(M^{\otimes p} \oplus L \otimes M^{\otimes p})$, since $P^1_{\mathbb{P}/Y}(\mathcal{O}_{\mathbb{P}}(1))$ is the pull-back of $M^{\otimes p} \oplus L \otimes M^{\otimes p}$ to $\mathbb{P}$ by [9, Proposition 6.3].

We will see that (2.3.1) comes from Y. Let τ be a local basis of L at $y \in Y$ and let s be one of M, and hence $\tau \otimes s^{\otimes p}$ is a local base of $L \otimes M^{\otimes p}$ at $y \in Y$. By Step 1, the dual $z \in L^\vee$ of τ can be seen as a local function on X, and hence $f := z^{lp}$ is one on Y. By Step 1, $f_{\geq 1} := f - f(y) \in \mathcal{O}_Y$ can be extended to a system of parameters $f_{\geq 1}, t_2, \ldots, t_n$ at $y \in Y$. By definition, it is easy to see that

$$z \otimes (\tau \otimes s^{\otimes p}) + dz \otimes (\tau \otimes s^{\otimes p}), \quad 1 \otimes (\tau \otimes s^{\otimes p}), \quad dt_i \otimes (\tau \otimes s^{\otimes p}) \ (i = 2, \ldots, n) \tag{2.3.2}$$

make a local basis of $P^1_X(\mathcal{O}_X(1))$ at every point x with $\phi(x) = y$ and that the first two make one of $P^1_{X/Y}(\mathcal{O}_X(1)) \cong \phi^*(M^{\otimes p} \oplus L \otimes M^{\otimes p})$.

We will look at the transition matrix of (2.3.1) with respect to these bases in (2.3.2). Let τ and τ' be local bases of L and s and s' ones of M with transition functions (τ/τ') and (s/s'), respectively. Let z and z' be the dual of τ and τ', respectively. Since

$$\begin{aligned} z \otimes (\tau \otimes s^{\otimes p}) + dz \otimes (\tau \otimes s^{\otimes p}) &= 1 \otimes (\tau \otimes s^{\otimes p})z = 1 \otimes (\tau' \otimes s'^{\otimes p})z\left(\frac{\tau}{\tau'}\right)\left(\frac{s}{s'}\right)^p \\ &= 1 \otimes (\tau' \otimes s'^{\otimes p})z'\left(\frac{s}{s'}\right)^p \\ &= \left(\frac{s}{s'}\right)^p \{z' \otimes (\tau' \otimes s'^{\otimes p}) + dz' \otimes (\tau' \otimes s'^{\otimes p})\}, \\ 1 \otimes (\tau \otimes s^{\otimes p}) &= 1 \otimes (\tau' \otimes s'^{\otimes p})\left(\frac{\tau}{\tau'}\right)\left(\frac{s}{s'}\right)^p \\ &= \left(\frac{\tau}{\tau'}\right)\left(\frac{s}{s'}\right)^p \otimes (\tau' \otimes s'^{\otimes p}) + \left(\frac{s}{s'}\right)^p d\left(\frac{\tau}{\tau'}\right) \otimes (\tau' \otimes s'^{\otimes p}), \end{aligned}$$

the entries of the transition matrix are functions over Y, and hence (2.3.1) is the pull-back of an exact sequence with a vector bundle E on Y fitting into

$$0 \to \left(\frac{\Omega^1_Y}{(L^\vee)^{\otimes lp}}\right) \otimes (L \otimes M^{\otimes p}) \to E \to (M^{\otimes p} \oplus L \otimes M^{\otimes p}) \to 0.$$

Finally, we will show that a^1 comes from Y. To this end, let $x \in X$ and we look at a^1 with respect to the basis $s_0^{\otimes p}, \ldots, s_r^{\otimes p}, v_0, \ldots, v_m$ of V in Step 2 and the local basis of $P_X^1(\mathcal{O}_X(1))$ at x in (2.3.2). By renumbering the indices if necessary, we may assume that $s_0(\phi(x)) \neq 0$ and $v_0(\phi(x)) \neq 0$. Then for a suitable affine open neighborhood U of $\phi(x)$ in Y, we have $M^{\otimes p}|U = \mathcal{O}_U s_0^{\otimes p}$ and $L \otimes M^{\otimes p}|U = \mathcal{O}_U v_0$ and hence $\tau := v_0 \otimes (s_0^{\vee})^{\otimes p}$ is a basis of $L|U$. Let z be the dual of τ. If $v_i = g_i v_0$ for $g_i \in \mathcal{O}_Y$ at $\phi(x)$, then we have

$$a^1(s_j^{\otimes p}) = 1 \otimes (\tau \otimes s_0^{\otimes p}) z \left(\frac{s_j}{s_0}\right)^p = \left(\frac{s_j}{s_0}\right)^p \{z \otimes (\tau \otimes s_0^{\otimes p}) + dz \otimes (\tau \otimes s_0^{\otimes p})\}$$

and

$$a^1(v_i) = 1 \otimes (\tau \otimes s_0^{\otimes p}) g_i = g_i \otimes (\tau \otimes s_0^{\otimes p}) + dg_i \otimes (\tau \otimes s_0^{\otimes p})$$

for $j = 0, \ldots, r$ and $i = 0, \ldots, m$. The all entries in the matrix with respect to the basis (2.3.2) are local functions on Y, and therefore a^1 is a pull-back of some $V \otimes \mathcal{O}_Y \to E$. This completes the proof of Step 3. $\square$

Step 4. (The image of $\iota^{(1)}$): *Assume that* $H^0(Y, L \otimes M^{\otimes p}) \to (L \otimes M^{\otimes p}) \otimes (\mathcal{O}_Y/m_y^3)$ *is surjective for every* $y \in Y$. (*This assumption holds true if* M *is sufficiently very ample.*) *Assume that the global sections* $s_0, \ldots, s_r \in H^0(M)$ *define an embedding of* Y in $\mathbb{P}^r$. *Then the induced morphism* $Y \to \mathbb{G}$ *in Step* 3 *is an embedding unless* $p = 2$ *and* $n = 2$.

Proof. We have only to show that the wedge $\varepsilon : \wedge^{n+1} V \otimes \mathcal{O}_Y \to \wedge^{n+1} E$ of the surjection $V \otimes \mathcal{O}_Y \to E$ in Step 3 defines an embedding of Y. To see that any distinct points y_1 and y_2 of Y are mapped to distinct points, by suitable change of bases of $H^0(M)$ and $H^0(L \otimes M^{\otimes p})$, we may assume that $s_0(y_1) \neq 0$, $s_1(y_1) = 0$, $v_0(y_1) \neq 0$, $s_0(y_2) \neq 0$, $s_1(y_2) \neq 0$, and $v_0(y_2) \neq 0$. Moreover, if $v_i = g_i v_0$ for $g_i \in \mathcal{O}_Y$ at y_2, we may assume that $g_1, \ldots, g_n$ make a system of parameters of $\mathcal{O}_Y$ at y_2 such that $g_2, \ldots, g_n$ can be extended to one of $\mathcal{O}_X$ at the points over y_2 and that $g_i \in m_{y_2}^2$ $(i \geq n)$. Then $\varepsilon(s_1^{\otimes p} \wedge v_0 \wedge v_2 \wedge \ldots \wedge v_n) \in \wedge^{n+1} E$ is zero at y_1 but nonzero at y_2, which means y_1 and y_2 are mapped to distinct points.

To see that the morphism is an immersion at a point y of Y, by suitable change of bases, we may assume that $v_0(y) \neq 0$ and $s_0(y) \neq 0$. Moreover if $v_i = g_i v_0$ at y for $g_i \in \mathcal{O}_Y$, we may assume that $t_1 := g_1, \ldots, t_n := g_n$ consist of a system of parameters of $\mathcal{O}_Y$ at y such that $t_2, \ldots, t_n$ can be extended to one of $\mathcal{O}_X$ at the points over y, and that $g_{n+i} - t_i^2 \in m_y^3$ $(i = 1, \ldots, n)$. If $n \geq 3$, we may assume furthermore that $g_{2n-2+i} - t_2 t_i \in m_y^3$ $(i = 3, \ldots, n)$. For τ and z in Step 3, set $\eta := (z \otimes (\tau \otimes s_0^{\otimes p}) + dz \otimes (\tau \otimes s_0^{\otimes p})) \wedge (1 \otimes \tau \otimes s_0^{\otimes p}) \wedge (dt_2 \otimes (\tau \otimes s_0^{\otimes p})) \wedge \cdots \wedge (dt_n \otimes (\tau \otimes s_0^{\otimes p}))$. Since

$$\varepsilon(s_0^{\otimes p} \wedge v_0 \wedge v_2 \wedge \cdots \wedge v_n) = \eta,$$
$$\varepsilon(s_0^{\otimes p} \wedge v_1 \wedge v_2 \wedge \cdots \wedge v_n) = t_1 \eta,$$
$$\varepsilon(s_0^{\otimes p} \wedge v_0 \wedge v_2 \wedge \cdots \wedge v_{i-1} \wedge v_{n+i} \wedge v_{i+1} \wedge \cdots \wedge v_n) - 2t_i \eta \in m_y^2 \eta \quad (i = 2, \ldots, n),$$

$$\varepsilon(s_0^{\otimes p} \wedge v_0 \wedge v_2 \wedge v_{2n+1} \wedge v_4 \wedge \cdots \wedge v_n) - t_2\eta \in m_y^2\eta,$$
$$\varepsilon(s_0^{\otimes p} \wedge v_0 \wedge v_{2n-2+i} \wedge v_3 \wedge \cdots \wedge v_n) - t_i\eta \in m_y^2\eta \quad (i = 3,\ldots,n) \text{ for } n \geq 3,$$

the morphism is an immersion at y unless $p = 2$ and $n = 2$, as required. □

3. Normal varieties of general type

Theorem 1.2 follows immediately from the following theorem.

Theorem 3.1. *Assume that $p > 2$. Let l be a positive integer not divisible by p. Let Y be a smooth projective variety of dimension $n \geq 2$. Let L be a very ample line bundle on Y and s a general global section of $L^{\otimes lp}$. Then there exist a normal projective variety X of dimension n with only isolated singularities, a finite morphism $\phi : X \to Y$, and an embedding $\iota : X \to \mathbb{P}^N$ satisfying the following properties*:
(1) *The canonical sheaf ω_X of X is isomorphic to $\phi^*(\omega_Y \otimes L^{\otimes lp-1})$.*
(2) *There exists a birational morphism σ: $\hat{X} \to X$ such that $\hat{X}$ is a smooth projective variety of general type with $\sigma^*\omega_X \subseteq \omega_{\hat{X}}$.*
(3) *The separable and inseparable degrees of ϕ are l and p, respectively.*
(4) *The Gauss map $\iota^{(1)} : \mathrm{Reg}(X) \to \mathbb{G} := \mathrm{Grass}(n, \mathbb{P}^N)$ of ι is the composite of $\phi|\mathrm{Reg}(X)$ and a morphism $\phi(\mathrm{Reg}(X)) \to \mathbb{G}$.*
(5) *If L is triple or more of a very ample line bundle, then the morphism $\phi(\mathrm{Reg}(X)) \to \mathbb{G}$ is an embedding.*

The proof of Theorem 3.1 proceeds in three steps.

Step 1. (Construction of X): *There exist a normal projective variety X with only isolated singularities, a finite morphism $\phi : X \to Y$ of separable and inseparable degree l and p, respectively, an $\mathcal{O}_Y$-linear $ds : (L^\vee)^{\otimes lp} \to \Omega_Y^1$, and an exact sequence*

$$0 \to \phi^*(L^\vee)^{\otimes lp} \xrightarrow{\phi^* ds} \phi^*\Omega_Y^1 \xrightarrow{d\phi} \Omega_X^1 \to \phi^*(L^\vee) \to 0.$$

Furthermore, the singular locus of X is exactly the inverse image $\phi^{-1}((ds)_0)$ of the zero locus of ds, and there exists a birational morphism $\sigma : \hat{X} \to X$ such that $\hat{X}$ is a smooth projective variety of general type with $\sigma^*\omega_X \subseteq \omega_{\hat{X}}$.

Proof. We follow the construction in [6]. The map $(L^\vee)^{\otimes lp} \to \mathcal{O}_Y$ induced by s defines an $\mathcal{O}_Y$-algebra $\mathscr{A} := \bigoplus_{i=0}^{lp-1}(L^\vee)^{\otimes i}$ with $H^0(Y, \mathscr{A}) = K$ and hence we have a connected closed subscheme $X = \mathbf{Spec}_Y\,\mathscr{A}$ of $\mathbb{V} := \mathbb{V}_Y(L^\vee)$ with finite morphism $\phi : X \to Y$. Let $\{U_i\}$ be an affine open covering of Y such that each $L|U_i$ is trivial with a basis τ_i. Let $z_i \in L^\vee$ be the dual of τ_i. If $s = f_i\tau_i^{lp}$ for $f_i \in \mathcal{O}_{U_i}$, then $\phi^{-1}(U_i)$ is defined by $z_i^{lp} - f_i$ in $\mathbb{V}_{U_i}(L^\vee|U_i)$. Since $df_i = (\tau_j/\tau_i)^{lp}\,df_j$, we have the $\mathcal{O}_Y$-linear map $ds : (L^\vee)^{\otimes lp} \to \Omega_Y^1$ mapping z_i^{lp} to df_i. Then ds is nonzero at $y \in Y$ if and only if X is smooth at every $x \in X$ with $y = \phi(x)$. The exact sequence is obtained by the same argument as in Step 1 in Section 2. If ds is zero at $y \in Y$, by [6, 18. Proposition], we have $f - (f(y) + t_1^2 + \ldots + t_n^2) \in m_y^3$ with $f(y) \neq 0$ for a suitable system of parameters $t_1,\dots,t_n$ at $y\in Y$ since s is general. This implies that at every $x\in X$ with $\phi(x)\in(ds)_0$, X has only isolated singularities given by a local equation $w^p-(t_1^2+\cdots+t_n^2)+h\in\mathcal{O}_{\mathbb{V}}$ for some $h\in m_x^3$ and a system of parameters $w,t_1,\dots,t_n$ at x of $\mathcal{O}_{\mathbb{V}}$. By blowing up of $\mathbb{V}$ at the singular points step by step (see [6, Section 21]) and by adjunction formula, we obtain a birational morphism $\sigma:\hat{X}\to X$ with smooth projective $\hat{X}$ and $\sigma^*\omega_X\subseteq\omega_{\hat{X}}$. Since $\omega_X\cong\phi^*(\omega_Y\otimes L^{\otimes lp-1})$ is ample, $\hat{X}$ is of general type. $\square$

Step 2: Set $V=H^0(Y,\mathcal{O}_Y\oplus L)$. *Let* $\iota:X\to\mathbb{P}^N:=\mathbb{P}(V)$ *be the composite*

$$X\to\mathbb{V}_Y(L^\vee)\to\mathbb{P}_Y(L^\vee\oplus\mathcal{O}_Y)\cong\mathbb{P}_Y(\mathcal{O}_Y\oplus L)\to\mathbb{P}(V).$$

Then the Gauss map $\iota^{(1)}:\mathrm{Reg}(X)\to\mathbb{G}$ *of* ι *factors through* $\phi:X\to Y$.

Proof. By the same argument as in Step 3 in Section 2, we see that the surjection $a^1:V\otimes\mathcal{O}_X\to P_X^1(\mathcal{O}_X(1))$ on $\mathrm{Reg}(X)$ comes from $Y\setminus(ds)_0$ by ϕ, and hence $\iota^{(1)}$ factors through ϕ. $\square$

Step 3: Assume that L is triple (or more) of some very ample line bundle. Then the morphism $Y\setminus(ds)_0\to\mathbb{G}$ *is an embedding.*

Proof. By the same argument as in Step 4 in Section 2, it turns out that $Y\setminus(ds)_0\to\mathbb{G}$ is an immersion. For the injectivity, we note that the n-plane in $\mathbb{P}^N$ corresponding to the image of $y\in Y$ is spanned by the tangent directions $(\Omega_Y^1/(L^\vee)^{\otimes lp})^\vee\otimes K(y)$ in $\mathbb{P}(H^0(L))\subset\mathbb{P}(V)$ and the ruling $\mathbb{P}((\mathcal{O}_Y\oplus L)\otimes K(y))\subset\mathbb{P}(V)$. Thus if distinct points y_1 and y_2 of $Y\setminus(ds)_0$ are mapped to a same point in $\mathbb{G}$, then the embedded tangent spaces to Y at y_1 and y_2 intersect in $\mathbb{P}(H^0(L))(\subset\mathbb{P}(V))$. By the following Lemma 3.2 together with our assumption, this is contradiction. $\square$

Lemma 3.2. *Let $Y\subset\mathbb{P}^L$ be a projective variety not contained in any hyperplane, defined over an algebraically closed field of arbitrary characteristic. Let $v_m:\mathbb{P}^L\to\mathbb{P}^M$ be the mth Veronese embedding. By $T_P(v_m(Y))$ we denote the embedded tangent space to $v_m(Y)$ at $P\in Y$ in $\mathbb{P}^M$. Then*

(1) *when $m=2$, if $Y\neq\mathbb{P}^L$, for any two general distinct points P,Q of Y, we have $T_P(v_2(Y))\cap T_Q(v_2(Y))=\emptyset$ in $\mathbb{P}^M$;*
(2) *when $m\geq 3$, for any two distinct smooth points P,Q of Y, we have $T_P(v_m(Y))\cap T_Q(v_m(Y))=\emptyset$ in $\mathbb{P}^M$.*

In particular, for the embedding ρ of Y into projective space by the complete linear system associated to $\mathcal{O}_{\mathbb{P}^L}(m)|Y$, if $m\geq 3$ then the embedded tangent spaces to $\rho(Y)$ do not intersect each other.

Proof. (1) For two distinct points $P,Q\in\mathbb{P}^L$, we may assume that $P=[1,0,\dots,0]$ and $Q=[1,z_1,\dots,z_L]$, where z_i are affine coordinates of $\mathbb{P}^L$. Since $\boldsymbol{v}:=v_2(Q)=[1,z_1,\dots,z_L,$

$\{z_iz_j\}_{0<i\le j\le L}] \in \mathbb{P}^M$, $T_Q(v_2(\mathbb{P}^L))$ is spanned by the row vectors of the matrix

$$\begin{bmatrix} \boldsymbol{v} \\ \dfrac{\partial \boldsymbol{v}}{\partial z_1} \\ \dfrac{\partial \boldsymbol{v}}{\partial z_2} \\ \vdots \\ \dfrac{\partial \boldsymbol{v}}{\partial z_L} \end{bmatrix} = \begin{bmatrix} 1 & z_1 & z_2 & \dots & z_L & \{z_iz_j\}_{0<i\le j\le L} \\ 0 & 1 & 0 & \dots & 0 & \left\{\dfrac{\partial z_iz_j}{\partial z_1}\right\}_{0<i\le j\le L} \\ 0 & 0 & 1 & \dots & 0 & \left\{\dfrac{\partial z_iz_j}{\partial z_2}\right\}_{0<i\le j\le L} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \left\{\dfrac{\partial z_iz_j}{\partial z_L}\right\}_{0<i\le j\le L} \end{bmatrix}.$$

Hence $T_P(v_2(\mathbb{P}^L))\cap T_Q(v_2(\mathbb{P}^L))$ corresponds to the solutions $[a_0, a_1, \dots, a_L]$ of the equation

$$a_0\boldsymbol{v} + a_1\frac{\partial \boldsymbol{v}}{\partial z_1} + \dots + a_L\frac{\partial \boldsymbol{v}}{\partial z_L} \in T_P(v_2(\mathbb{P}^L)).$$

By Euler's formula,

$$[a_0, a_1, \dots, a_L] = [2, -z_1, \dots, -z_L] \tag{3.2.1}$$

is a solution. If $z_i \neq 0$, the submatrix consisting of column vectors corresponding to $z_1z_i, z_2z_i, \dots, z_Lz_i$ is of rank L or more in any characteristic. Since $P \neq Q$ we have $z_i \neq 0$ for some i, and hence (3.2.1) is a unique solution up to scalar multiple. Therefore $T_P(v_2(\mathbb{P}^L)) \cap T_Q(v_2(\mathbb{P}^L)) = \{\boldsymbol{u} = [2, z_1, \dots, z_L, 0, \dots, 0] \in \mathbb{P}^M\}$.

For a given $P \in Y$, up to $\mathrm{PGL}(\mathbb{P}^L)$, we may assume that

$$T_P(v_2(Y)) \subseteq T_P(v_2(\mathbb{P}^L)) \cap V(W_L) \subset T_P(v_2(\mathbb{P}^L)),$$

where W_L is a homogeneous coordinate of $\mathbb{P}^M$. For general $Q \in Y$ with $Q \notin V(Z_0) \subset \mathbb{P}^L$, if $T_P(v_2(Y)) \cap T_Q(v_2(Y)) \neq \emptyset$, then this is a one point set $\{\boldsymbol{u}\} \subset V(W_L)$, and hence $z_L = 0$. Since $Q \in Y$ is general, Y is contained in the hyperplane $Z_L = 0$, contradiction.

(2) In the same way, we set $P = [1, 0, \dots, 0]$, $Q = [1, z_1, \dots, z_L] \in \mathbb{P}^L$, and

$$\boldsymbol{v} := v_m(Q) = [1, z_1, \dots, z_L, \{z_iz_j\}_{0<i\le j\le L}, \{z_iz_jz_k\}_{0<i\le j\le k\le L}, \dots].$$

We look at the solutions $[a_0, a_1, \dots, a_L]$ of the equation

$$\boldsymbol{w} := a_0\boldsymbol{v} + a_1\frac{\partial \boldsymbol{v}}{\partial z_1} + \dots + a_L\frac{\partial \boldsymbol{v}}{\partial z_L} \in T_P(v_m(\mathbb{P}^L)).$$

By (1), $[a_0, a_1, \dots, a_L] = [2, -z_1, \dots, -z_L]$ is the only candidate of the solution. Since

$$2z_i^3 - \left(z_1\frac{\partial z_i^3}{\partial z_1} + \dots + z_L\frac{\partial z_i^3}{\partial z_L}\right) = -z_i^3$$

the entry corresponding to z_i^3 of $\boldsymbol{w}$ is nonzero if $z_i \neq 0$, and hence $\boldsymbol{w} \notin T_P(v_m(\mathbb{P}^L))$ if $Q \neq P$. This implies $T_P(v_m(\mathbb{P}^L))\cap T_Q(v_m(\mathbb{P}^L))=\emptyset$. Therefore $T_P(v_m(Y))\cap T_Q(v_m(Y))=\emptyset$ for $P \neq Q \in \mathrm{Reg}(Y)$.

The final part follows from (2) since v_m is obtained from ρ by taking the linear projection and the linear inclusion corresponding to $H^0(Y, \mathscr{O}_{\mathbb{P}^L}(1)|Y)^{\otimes m} \to H^0(Y, \mathscr{O}_{\mathbb{P}^L}(m)|Y)$. $\square$

4. $\mathbb{P}^n$

To prove Theorem 1.3, we need the following lemma.

Lemma 4.1. *Let $\pi : \mathbb{P}_Y(G) \to Y$ be a projective bundle over an n-dimensional projective variety Y associated with a locally free sheaf G of rank $n+1$ on Y. Let X be an n-dimensional smooth projective subvariety of $\mathbb{P}_Y(G)$ such that the composite $\phi : X \to \mathbb{P}_Y(G) \to Y$ is a finite morphism factoring through the K-Frobenius of X. Let $\iota : X \to \mathbb{P}^N$ be the embedding given by an embedding $\mathbb{P}_Y(G) \to \mathbb{P}^N$ as scroll,* (*i.e.*, *every fiber of π is linear in $\mathbb{P}^N$*), *whose image is not contained in any hyperplane. Then the Gauss map $\iota^{(1)}$ of ι is the composite of ϕ and an embedding $Y \to \mathbb{G} = \mathrm{Grass}(n, \mathbb{P}^N)$.*

Proof. By definition of bundles of principal parts (see for example [9, Section 6]), we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & \Omega^1_{\mathbb{P}/Y} \otimes \mathscr{O}_{\mathbb{P}}(1)|X & \to & \pi^* G|X & \to & \mathscr{O}_{\mathbb{P}}(1)|X & \to & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \to & \Omega^1_{\mathbb{P}/Y} \otimes \mathscr{O}_{\mathbb{P}}(1)|X & \to & P^1_{\mathbb{P}/Y}(\mathscr{O}_{\mathbb{P}}(1))|X & \to & \mathscr{O}_{\mathbb{P}}(1)|X & \to & 0 \\
 & & \downarrow\beta & & \downarrow & & \| & & \\
0 & \to & \Omega^1_{X/Y} \otimes \mathscr{O}_X(1) & \to & P^1_{X/Y}(\mathscr{O}_X(1)) & \to & \mathscr{O}_X(1) & \to & 0 \\
 & & \uparrow\alpha & & \uparrow & & \| & & \\
0 & \to & \Omega^1_X \otimes \mathscr{O}_X(1) & \to & P^1_X(\mathscr{O}_X(1)) & \to & \mathscr{O}_X(1) & \to & 0.
\end{array}$$

The map α is an isomorphism, since ϕ factors through the K-Frobenius by assumption. Also β is isomorphism, since β is a surjective map between locally free sheaves of same rank. Therefore the middle terms are isomorphic.

For $V := H^0(\mathscr{O}_{\mathbb{P}^N}(1))$, it is easy to see that $a^1 : V \otimes \mathscr{O}_X \to P^1_X(\mathscr{O}_X(1))$ is the pull-back of the canonical surjection $V \otimes \mathscr{O}_Y \to G$ by ϕ under the isomorphism $\phi^* G \cong P^1_X(\mathscr{O}_X(1))$. By assumption and Lemma 4.2 below, the canonical surjection defines an embedding $Y \to \mathbb{G}$, and therefore the image of $\iota^{(1)}$ is Y. $\square$

Lemma 4.2. *Let Y be a possibly singular projective variety of dimension n and G a locally free sheaf of rank $r+1$. Let $V \subset H^0(Y, G)$ be a K-vector space generating G and $\varepsilon : V \otimes \mathscr{O}_Y \to G$ the corresponding surjection. Assume that the morphism $\mathbb{P}_Y(G) \to \mathbb{P}(V)$ defined by V is an embedding. Then the morphism $Y \to \mathrm{Grass}(r, \mathbb{P}(V))$ defined by ε is also an embedding.*

Proof. We will check that the image of $\bigwedge^{r+1} V$ to global sections of $\bigwedge^{r+1} G$ separates two points and tangent directions. Since the separation of two points are clear, we have only to check the separation of tangent directions at a point P of $\mathbb{P} := \mathbb{P}_Y(G)$ over $y \in Y$. Around $y \in Y$, G is trivialized as $G = \mathcal{O}e_0 \oplus \cdots \oplus \mathcal{O}e_r$ so that $P = [1, 0, \ldots, 0] \in \mathbb{P}(G \otimes K(y))$. Our assumption implies the surjectivity of the map

$$V \to \left(\frac{\mathcal{O}_Y}{m_y^2}\right) e_0 \oplus K(y)e_1 \oplus \cdots \oplus K(y)e_r$$

since the vector space in the right-hand side is isomorphic to $\mathcal{O}_{\mathbb{P}}(1) \otimes (\mathcal{O}_{\mathbb{P}}/m_P^2)$. Let $t_1, \ldots, t_d$ be lifts of basis of m_y/m_y^2 in $\mathcal{O}_Y$ and let $v_0, \ldots, v_r, v_{01}, \ldots, v_{0d}$ be elements of V such that $\varepsilon(v_i) - e_i$ and $\varepsilon(v_{0j}) - t_j e_0$ $(i = 0, \ldots, r;\ j = 1, \ldots, d)$ are contained in $m_y^2 e_0 \oplus m_y e_1 \oplus \cdots \oplus m_y e_n$. Since

$$\wedge\varepsilon(v_0 \wedge v_1 \wedge \cdots \wedge v_r) - e_0 \wedge \cdots \wedge e_r \in m_y e_0 \wedge \cdots \wedge e_r,$$

$$\wedge\varepsilon(v_{0i} \wedge v_1 \wedge \cdots \wedge v_r) - t_i e_0 \wedge \cdots \wedge e_r \in m_y^2 e_0 \wedge \cdots \wedge e_r,$$

$\wedge^{r+1} V \to \wedge^{r+1} G \otimes (\mathcal{O}_Y/m_y^2)$ is surjective, as required. $\square$

Proof of Theorem 1.3. The idea comes from the curve case in [2,3] and [10, (2.13)]. Let L be a very ample line bundle on $\mathbb{P}^n$ such that $\omega_{\mathbb{P}^n} \otimes L^{\otimes p-1}$ is ample. Let s be a general global section of $L^{\otimes p}$. By Step 1 in Section 3 with $l = 1$, we have the finite morphism $\psi : Y \to \mathbb{P}^n$ from a normal projective variety Y to $\mathbb{P}^n$ associated with s, which corresponds to $\phi : X \to Y$ in Step 1 in Section 3 with the notation there. By construction, Y is of general type. By looking at ψ locally, there exists a purely inseparable morphism $\pi : \mathbb{P}^n \to Y$ such that $\psi \circ \pi$ is the K-Frobenius morphism F of $\mathbb{P}^n$. Take a finite morphism $f : \mathbb{P}^n \to \mathbb{P}^n$ and set $\phi = \pi \circ f \circ F$. Let $\Gamma_\phi(\cong \mathbb{P}^n) \subset \mathbb{P}^n \times Y$ be the graph of ϕ. Embed $\mathbb{P}^n \times Y$ in a suitable projective space $\mathbb{P}^N$ as a scroll, not contained in any hyperplane. Let $\iota : \mathbb{P}^n \cong \Gamma_\phi \subset \mathbb{P}^n \times Y \subset \mathbb{P}^N$ be the embedding. By Lemma 4.1, $\iota^{(1)}$ factors through ϕ with $\iota^{(1)}(\mathbb{P}^n) = Y$, as required. $\square$

Example. With the same notation as in proof of Theorem 1.3, we write down the embedding ι in Theorem 1.3 explicitly, when $n = 2$ and $p \geq 5$ for simplicity. Let Y_0, Y_1, Y_2 be homogeneous coordinates of $\mathbb{P}^2$. Set $L = \mathcal{O}_{\mathbb{P}^2}(1)$ so that $\omega_{\mathbb{P}^2} \otimes L^{\otimes p-1} = \mathcal{O}_{\mathbb{P}^2}(p-4)$ is ample. Let $S(Y_0, Y_1, Y_2) = \sum a_{i_0 i_1 i_2} Y_0^{i_0} Y_1^{i_1} Y_2^{i_2}$ be a general global section of $\mathcal{O}_{\mathbb{P}^2}(p)$ with $a_{i_0 i_1 i_2} \in K$ and set $S_1(Y_0, Y_1, Y_2) = \sum a_{i_0 i_1 i_2}^{1/p} Y_0^{i_0} Y_1^{i_1} Y_2^{i_2}$. The (geometric) line bundle $\mathbb{V}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(-1)) \to \mathbb{P}^2$ is realized by the linearprojection $\mathbb{P}^3 \backslash Q \to \mathbb{P}^2$, $[W_0, W_1, W_2, W_3] \mapsto [W_0, W_1, W_2]$ with center $Q = [0, 0, 0, 1]$, since

$$\mathbb{V}(\mathcal{O}_{\mathbb{P}^2}(-1)) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}) \backslash \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(-1)) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)) \backslash \mathbb{P}(\mathcal{O}_{\mathbb{P}^2})$$

and since $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ can be seen the blowing up of $\mathbb{P}^3$ at Q with the exceptional set $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2})$, or the graph of the linear projection $\mathbb{P}^3 \backslash Q \to \mathbb{P}^2$ with center Q. Under the identification, Y is defined by $W_3^p - S(W_0, W_1, W_2) = 0$ in $\mathbb{P}^3$ and $\psi : Y \to \mathbb{P}^2$ is induced by the linear projection. Then $\pi : \mathbb{P}^2 \to Y$ in the proof is induced by the morphism $\mathbb{P}^2 \to \mathbb{P}^3$ given by $[Y_0, Y_1, Y_2] \mapsto [Y_0^p, Y_1^p, Y_2^p, S_1(Y_0, Y_1, Y_2)]$, since this factors through Y and the composite $\mathbb{P}^2 \to \mathbb{P}^2$ with ψ is the K-Frobenius F of $\mathbb{P}^2$. Let $f : \mathbb{P}^2 \to \mathbb{P}^2$

be the double cover defined by $[Y_0, Y_1, Y_2] \mapsto [Y_0^2, Y_1^2, Y_2^2]$. Then $\phi := \pi \circ f \circ F$ is given by $[Y_0, Y_1, Y_2] \mapsto [Y_0^{2p^2}, Y_1^{2p^2}, Y_2^{2p^2}, S_1(Y_0^{2p}, Y_1^{2p}, Y_2^{2p})]$. Take the embedding ι to be $\mathbb{P}^2 \cong \Gamma_\phi \hookrightarrow \mathbb{P}^2 \times Y \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}^{11}$, where the final inclusion is the Segre embedding. Thus ι is given by

$$[Y_0, Y_1, Y_2] \mapsto \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \end{bmatrix} [Y_0^{2p^2} \quad Y_1^{2p^2} \quad Y_2^{2p^2} \quad S_1(Y_0^{2p}, Y_1^{2p}, Y_2^{2p})]$$

and the image of $\iota^{(1)}$ is isomorphic to Y.

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